FLEXURAL VIBRATION OF A PLATE WITH ELASTIC

ROTATIONAL CONSTRAINT ON BOUNDARY

by

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and

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NOMENCLATURE

W = deflection

ρ = mass density

h = thickness

t = time

E = Young's Modulus

η = Poisson's ratio

$D = \dfrac{Eh^3}{12(1-\eta^2)}$ = flexural rigidity of plate

F = transverse load

n = coordinate in normal direction at plate boundary

s = coordinate in tangential direction at plate boundary

α = rotational compliance or stiffness

I = moment of inertia of cross-section of bar

A = Area of cross-section of bar

$\gamma = EI$

$\nu = \dfrac{\gamma}{\alpha \ell}$

$\theta = K_m \ell$

K_n = frequency number of nth mode of vibration

R = ratio of depth of notch to thickness of plate or of bar

$P = \dfrac{C}{a^2}\sqrt{\dfrac{D}{\rho h}}$ for plate

$P = \dfrac{C}{\ell^2}\sqrt{\dfrac{Eh^2}{12\rho}}$ for bar

P = circular frequency of fundamental mode

C = Constant

a = length of side of plate

ℓ = length of bar

INTRODUCTION

In technology it is sometimes necessary to investigate experimentally the effects of damping materials, stiffeners, or even thickness variations on the vibration characteristics of plates. An especially interesting example of the need for obtaining simply supported rectangular plates occurred in a study by T. Vogel of the sound-insulating properties of single walls and double-walled partitions(1)[1]. In order to make such investigations, mathematically specified boundary conditions must usually be experimentally obtainable. This need is particularly felt if comparison is to be made between theoretical predictions and experimental results.

Providing appropriate boundary conditions for flexural vibration experiments with plates offers a challenge to the researcher. The problem is much more difficult than that for the flexural vibration of thin bars, which are practically one-dimensional in space variables. The so-called fixed plate may be obtained by welding its boundary to a rigid support or by machining the plate from a solid block of material leaving a thick ring at the boundary which may be heavily clamped to a ridgid support. On the other hand, the condition of zero rotational constraint or what is sometimes called simple support is much more difficult to obtain. There are also the cases of intermediate rotational constraint which arise in connection with various physical problems.

It is the purpose of this paper to describe a method of suitably providing given boundary conditions for bending experiments with thin elastic plates. The case of zero rotational constraint and zero deflection at the boundary is especially studied for use in a future investigation of the dynamical characteristics of certain structures. The investigation is limited to small vibrations; that is, deflections are small compared to the thickness of the plate.

[1] numbers in parentheses refer to bibliography at end of paper.

ANALYTICAL DEFINITION OF THE PROBLEM

The bending theory of thin elastic plates is well known by now (2). In Fig. 1 is shown diagrammatically a curvilinear plate. The differential equation for the forced transverse vibrations of such a plate may be written as follows:

$$D \nabla^4 W + \rho h \frac{\partial^2 W}{\partial t^2} = F - - - - - - - - - [1]$$

The boundary conditions studied in the present paper are:

$$W = 0$$
$$D \left(\frac{\partial^2 W}{\partial n^2} + \eta \frac{\partial^2 W}{\partial s^2} \right) = \alpha \frac{\partial W}{\partial n} \Bigg\} - - - - - - [2]$$

The latter condition in [2] states that the bending moment at the edge is proportional to the normal derivative or slope at the edge. The essential problem is to see how this boundary condition can be met in the contemplated experiments with plates. It will be shown that a V-notch groove near and parallel to the boundary, as shown in Fig. 1(b), will provide the means.

It may be here noted that the general solution of [1] subject to [2] is difficult and not available in the literature.

PRELIMINARY STUDY OF FLEXURAL VIBRATIONS OF A BAR

Because it is relatively simple to analytically solve the flexural vibration problem for a long thin bar and to make experiments with it, a preliminary study of the effect of V-notches was conducted for the bar. The essential question is whether a notch can be machined near the end of the bar to provide a given rotational constraint at that end.

For the small flexural vibrations of a bar, the differential equations and boundary conditions are (3):

$$- EI \frac{\partial^4 W}{\partial x^4} + \rho A \frac{\partial^2 W}{\partial t^2} = F(x,t) - - - - - - [3]$$

The boundary conditions are similar to those for the plate:

$$W = 0$$
$$EI \frac{\partial^2 W}{\partial x^2} = \alpha \frac{\partial W}{\partial x} \Bigg\} - - - - - - [4]$$

2
See nomenclature table for definition of symbols.

If $F \equiv 0$, a solution of $\lfloor 3 \rfloor$ subject to $\lfloor 4 \rfloor$ gives the following frequency equation:

$$(1- \cos\theta\cosh\theta)+ 2\nu\theta(\cosh\theta\sin\theta - \sinh\theta\cos\theta) + 2\nu^2\theta^2\sin\theta\cosh\theta = 0 \quad [5]$$

The values θ which satisfy $[5]$ for a given ν provide all of the eigen values of the problem, however in the subsequent work comparisons between experiment and theory are made only for the fundamental frequency.

The basic assumption in the investigation of the rotational compliance or constraint at the boundary is that the fundamental frequency determines it . In other words an experimental determination of the fundamental frequency by the resonance method is made. In the case of the bar, however, it was also rather easy to measure the stiffness factor ν in equation $[5]$ by means of statical loading tests and compute frequencies corresponding to these stiffnesses.

EXPERIMENTAL INVESTIGATION OF BAR

While the main interest of this investigation centered in the plate problem a few notched bars were prepared and tested. Cold rolled flat bar stock 3/8" x 3/16" in cross-section were cut into bars of appropriate length and notched near the ends as shown in Figs. 2a and 4. The symbolical equivalent of these bars is shown in Fig. 2b. For a given spring stiffness or constraint at the end of the bar a given boundary condition corresponding to $\lfloor 4 \rfloor$ is obtained. In order to statically determine the rotational spring constraints or compliances, bending tests were performed on the bars held rigidly just outside of the notches with apparatus shown in Fig. 3. For a series of increasing loads W on the free end of a bar the corresponding rotations near the supported end were measured with a micrometer as indicated. Contact of the micrometer with the rotation angle indicator was made apparent by an electron tube device which is quite sensitive for statical work of this type. Four small bars each with a 90° V-notch of given depth were tested. From the data of these tests, angular rotation as function of

moment was plotted and the slope of the resulting approximate straight line was determined for each notch depth. The moment per unit angular deflection α , was thus determined and plotted in Fig. 7 as a function of the ratio of notch depth to bar thickness, R. The angular compliance is the constant required in boundary condition [4] of the analytical formulation.

The apparatus for determining the frequencies of vibration of the bar as well as of the plate is shown in Fig. 5 and 6. The ends of the bar were clamped rigidly in the frame leaving the notched portions just clear of the clamps as shown in Fig. 5. The electromagnet, shown in Fig. 4, driven by a Hewlett-Packard oscillator was used to excite resonant vibrations in the bars. Resonance was determined with a crystal type vibration pick-up held lightly against the surface of the bar. The output from the pickup was fed directly to a cathode ray oscilloscope where the resonance condition could be readily detected.

The variation of θ , root of frequency equation [5], with stiffness factor ν , is shown as a curve in Fig. 8. Four experimental points for the fundamental mode are shown here also. The value ν was computed from the experimentally determined quantities and θ was obtained from measured vibration frequencies for given notch depth which corresponded with experimentally determined rotational compliance α .

Finally the experimental results are plotted in Fig. 9 showing the relation of physical constant C in circular frequency equation with notch depth ratio R. The fundamental frequencies for R equal zero and R equal unity are calculated from theory of fixed-fixed and hinged-hinged bar. The intermediate four prints were determined from measured frequency and notch ratio R. The curve was drawn through the two theoretical end points and fitted to the four experimental points.

EXPERIMENTAL INVESTIGATION OF PLATE

For the purpose of investigating plates with grooved rims approximating definite rotational edge constraints, flat square plates of steel 5 3/4" x 5 3/4"

were grooved as shown in Fig. 1. A plate with a given notch ratio R was then clamped at the outer edge in rigid frame shown in Fig. 6. A resonance test was performed just as described for the bar.

The experimental results are plotted in Fig. 10 showing the relation of physical constant C in circular frequency equation with notch length ratio R. The fundamental frequencies for R equal to zero and R equal to unity are taken from the theoretical solutions of S. Iguchi (4). The intermediate four experimental points were determined from measured frequency and notch ratio R. The curve was drawn through the two theoretical end points and fitted to the four experimental points.

DISCUSSION

Although the theory mentioned in the first part of this paper is for thin curvilinear plates in general, the experiments were made with square plates. It would seem that tests on a square plate provide a fair check on the idea for using notches to produce any desired rotational boundary constraint on plates in general. Also, the square plate with hinged edges is now actually required in an independent investigation of certain dynamical problems involving plates.

It is considered that a study of the fundamental frequency of vibration of a grooved plate held rigidly on the rim just outside of the groove will give a suitable test of the elastic rotational constraint on its boundary.

A theoretical solution for arbitrary rotational edge constraint of plates does not exist in the literature. However, a solution for the elastic rectangular plate with various edge conditions, including the clamped one is given in an important paper by Iguchi (4). The modification of Iguchi's theory to include the arbitrary elastic constraints can be carried out but the extent of the analysis was considered to be too time consuming for the purposes of the present investigation. If the work were accomplished a curve for the plate analogous to that for the bar shown in Fig. 8 could be constructed.

Of course, it is not being suggested in this paper that sharp V-notches be used in structural designs. Obviously damaging fatigue stress conditions may at times be produced in this manner. However, for certain test purposes the V-notch is inexpensive and convenient. In the present tests, no difficulties were experienced. It is obvious that if a smaller stress raiser is desired at the groove, other reduced-section designs may be employed at the notch.

As mentioned in the introduction, the special case of a simply supported plate is a very useful one in connection with theoretical investigation of plates involving damping materials, sound reducing partitions, and effectiveness of stiffener systems. The latter problem is now being investigated by the senior author, with plates requiring zero rotational constraint and zero bending deflection. Since the actual notch depth R equal to unity is structurally impossible if the continuity of the plate is to be preserved it is fortunate that the curve of C, R in Fig. 10 approaches asymptotically to a horizontal line through the theoretical value of C at R equal to unity. This being the case, it would seem that any notch ratio from eight-tenths to unity would give a fair representation of a simply supported plate. Even though the notch is deep in this case, the amount of material remaining to take the shear load and maintain the deflection practically zero is ample.

An attempt to produce simple support of a plate by use of rollers placed along the edge was made but adjustment of holding screws to maintain the rollers in place was quite indeterminate and gave widely varying frequencies for different adjustments.

The condition of simple support for thin bars can best be obtained by ball bearing supports 5 . . Such an arrangement also minimizes stretch of the middle surface. The somewhat higher experimental frequencies of bars tested by the present method of notches shows probably some effect of stretch of the middle surface although slight.

CONCLUSIONS

An experimental method utilizing a grooved rim has been devised for producing a thin elastic plate having any prescribed rotational edge constraint.

In particular, by the method the frequently useful case of zero rotational edge constraint and zero deflection can be obtained as a limiting case by making the notch depth ratio not actually unity but some value between three-quarters and unity.

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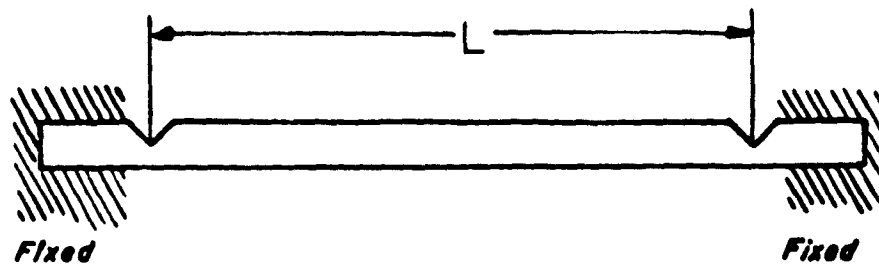
(a)   Curvilinear plate showing normal and tangent at point on boundary.

Groove

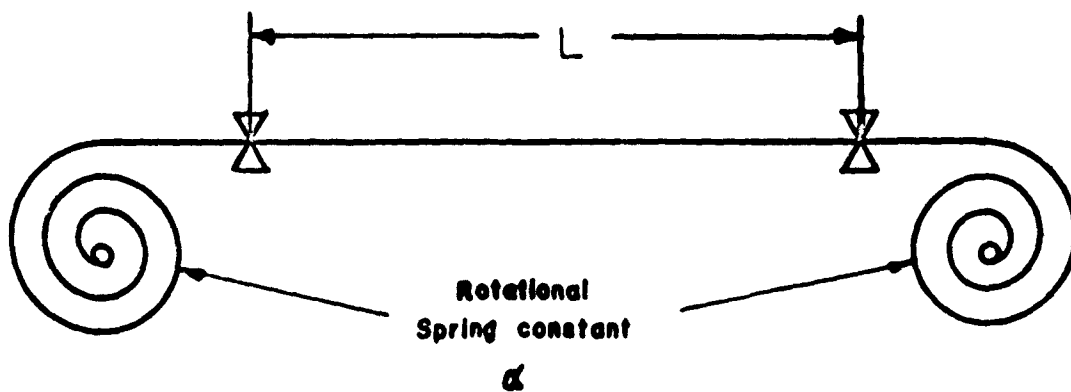Fixed

Fixed

Section A-A
Showing notched groove
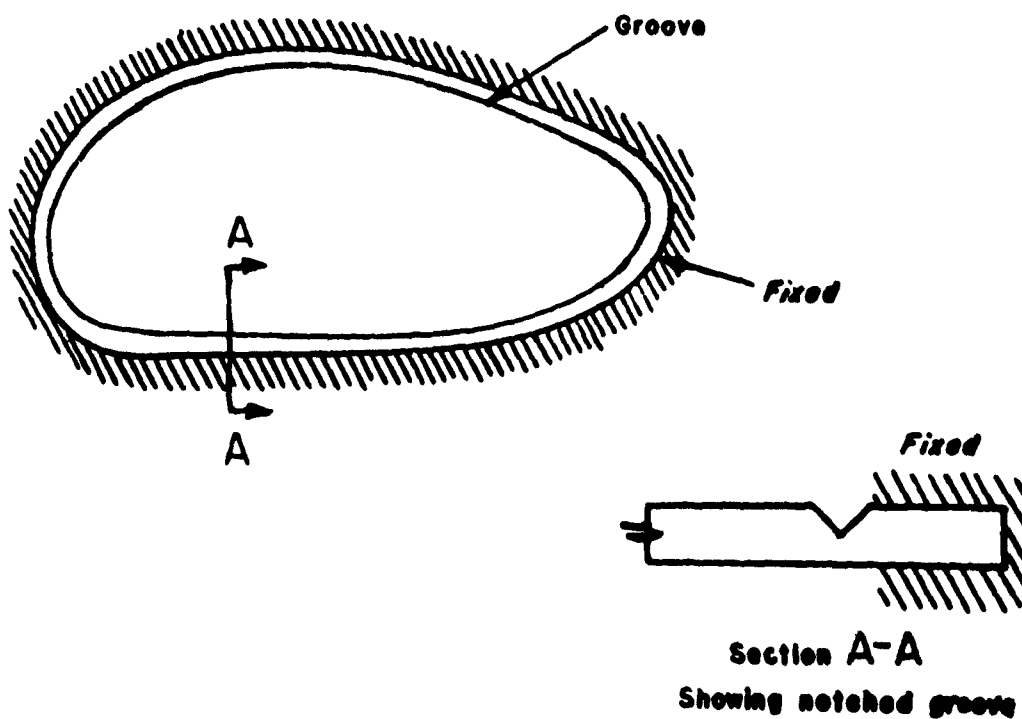
(b)   Device for providing rotational constraint along boundary.

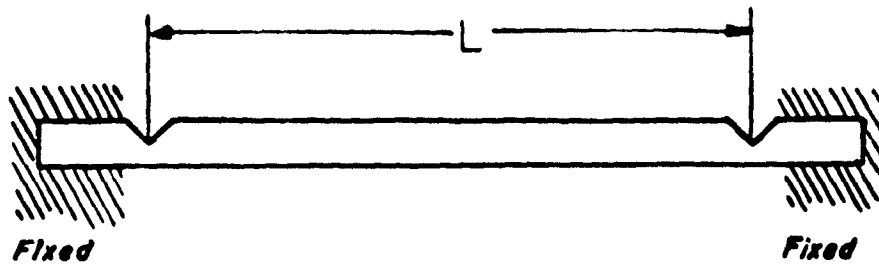Figure 1 Curvilinear flat plate with elastic rotational constraint along boundary.

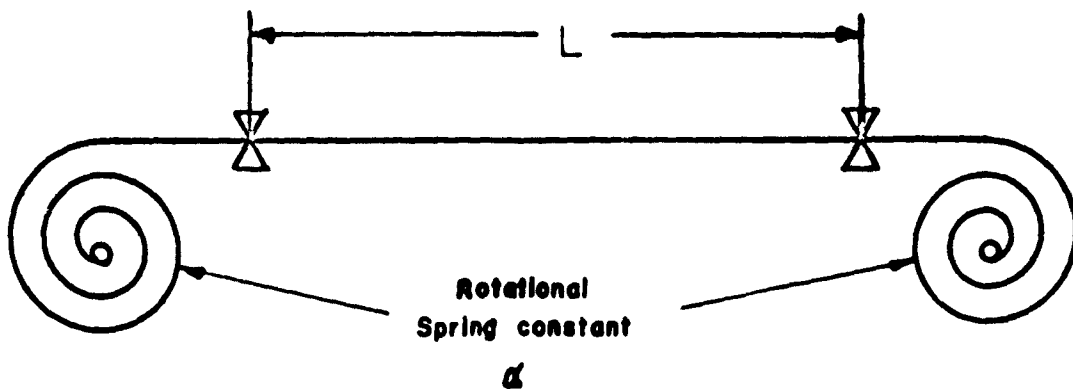(a)   Scematic arrangement of fixed ends with adjacent notches.

(b)   Symbolically equivalent bar with rotational end constraints and with zero deflection at the ends.
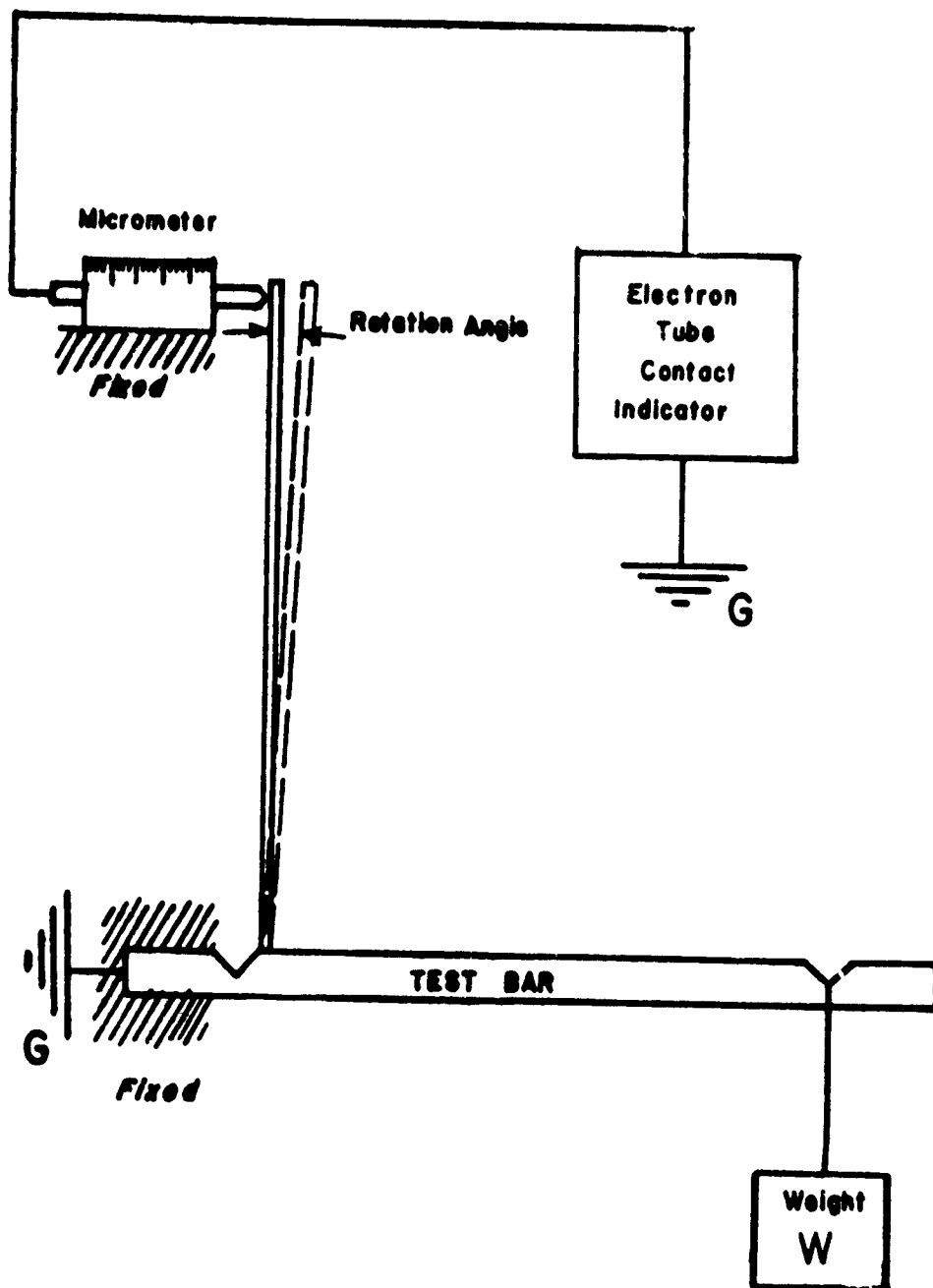
Figure 2   Bar with notched ends.

(a)   Scematic arrangement of fixed ends with adjacent notches.

Rotational
Spring constant
α

(b)   Symbolically equivalent bar with rotational end constraints and with zero deflection at the ends.

Figure 2   Bar with notched ends.

Figure 3   Apparatus for determination of the rotational compliance at end of bar.
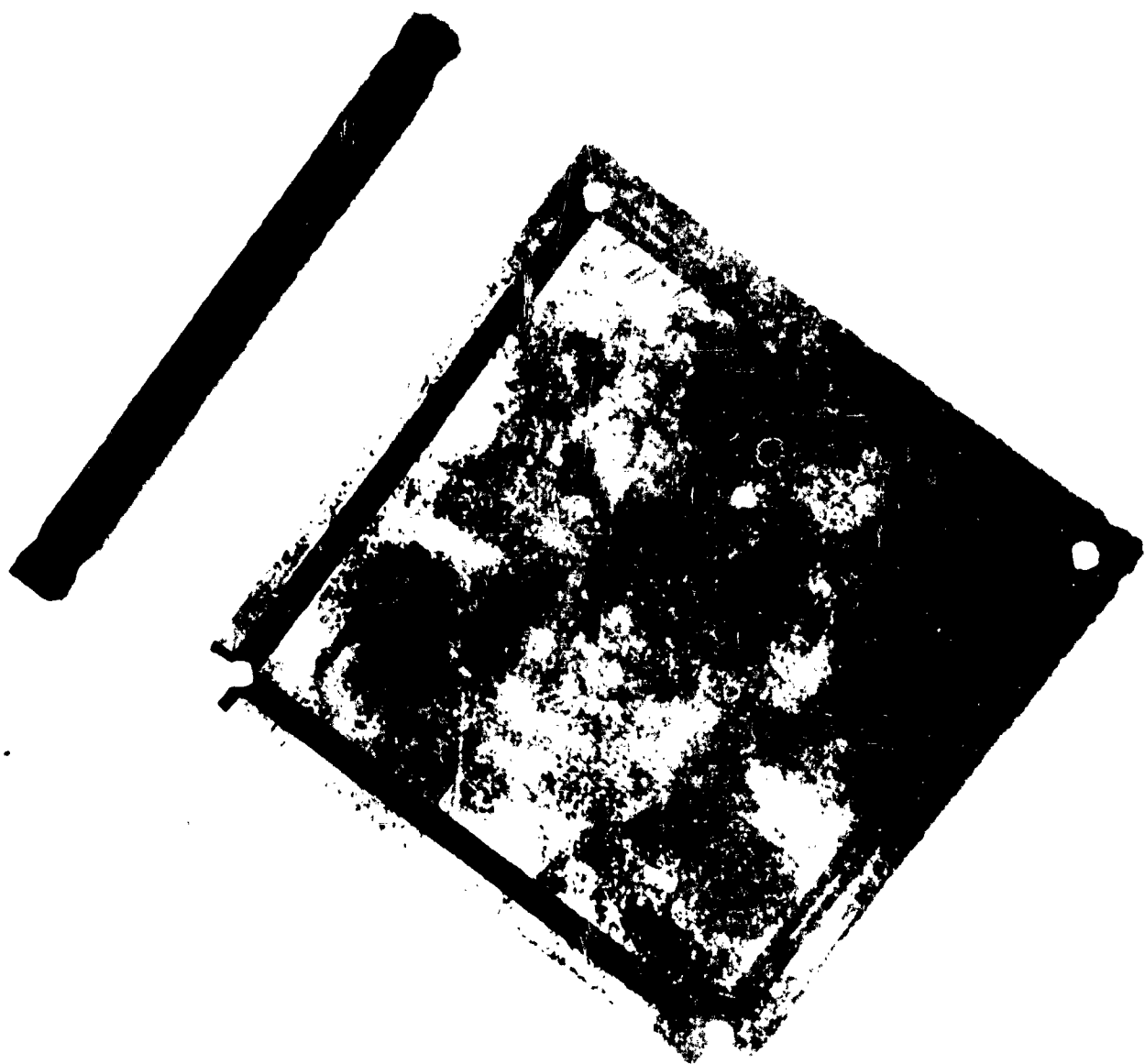
FIGURE 4 NOTCHED BAR AND NOTCHED PLATE USED IN EXPERIMENTS.

FIG. 16-3. DESIGN A 3/4 #3 VIBRATION EXPERIMENTS SHOWING SUPPORT FRAME AND ELECTROMAGNET.
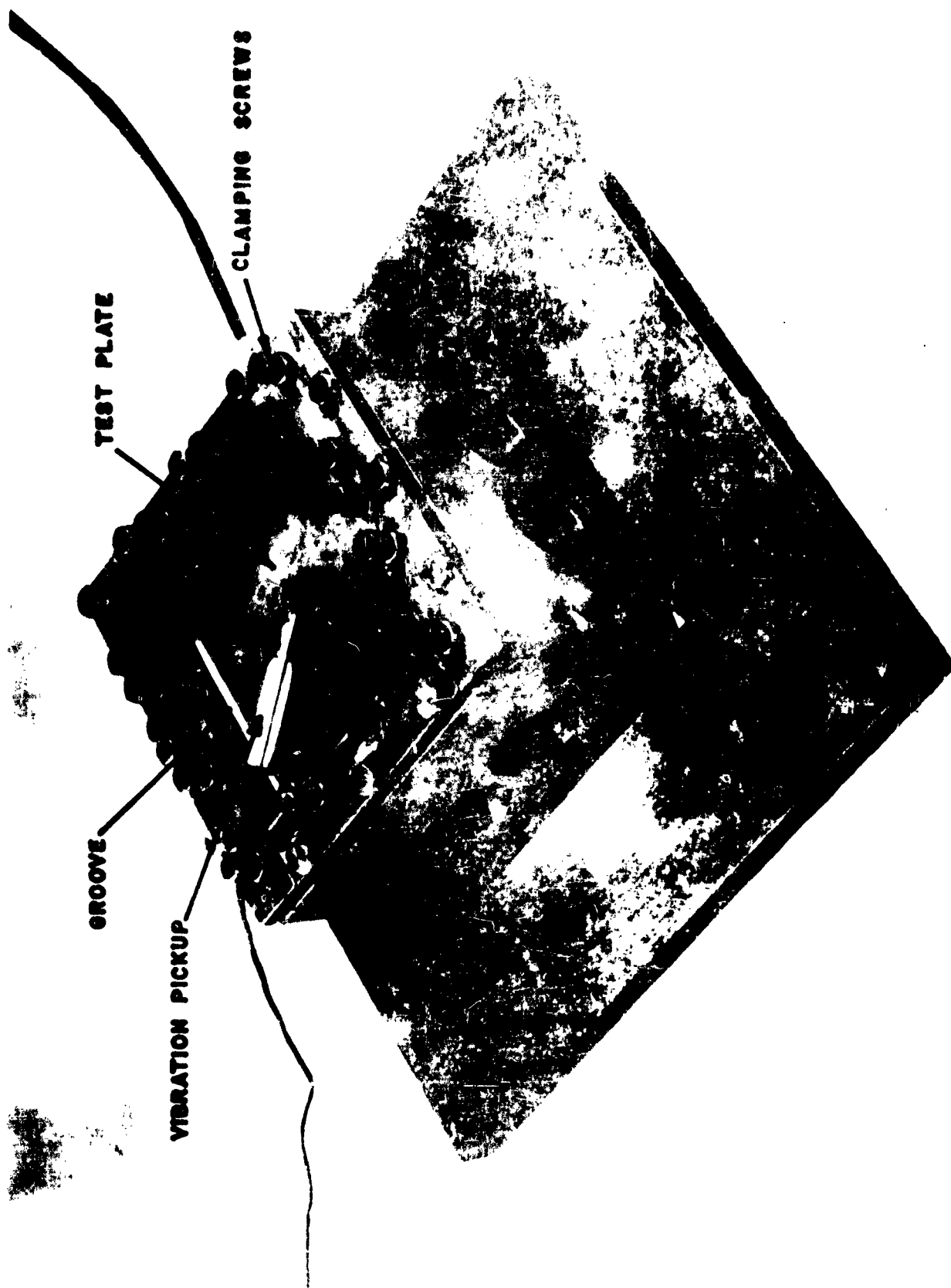
CLAMPING SCREWS

TEST PLATE

GROOVE

VIBRATION PICKUP

FIGURE 6 ASSEMBLED APPARATUS FOR VIBRATION EXPERIMENTS ON PLATE SHOWING VIBRATION PICKUP

O--- Experimental data

α in inch lbs./radian

1.5 ×10⁴

1.0 ×10⁴

.5 ×10⁴

0

R

.1   .2   .3   .4   .5   .6   .7   .8   .9   1.0
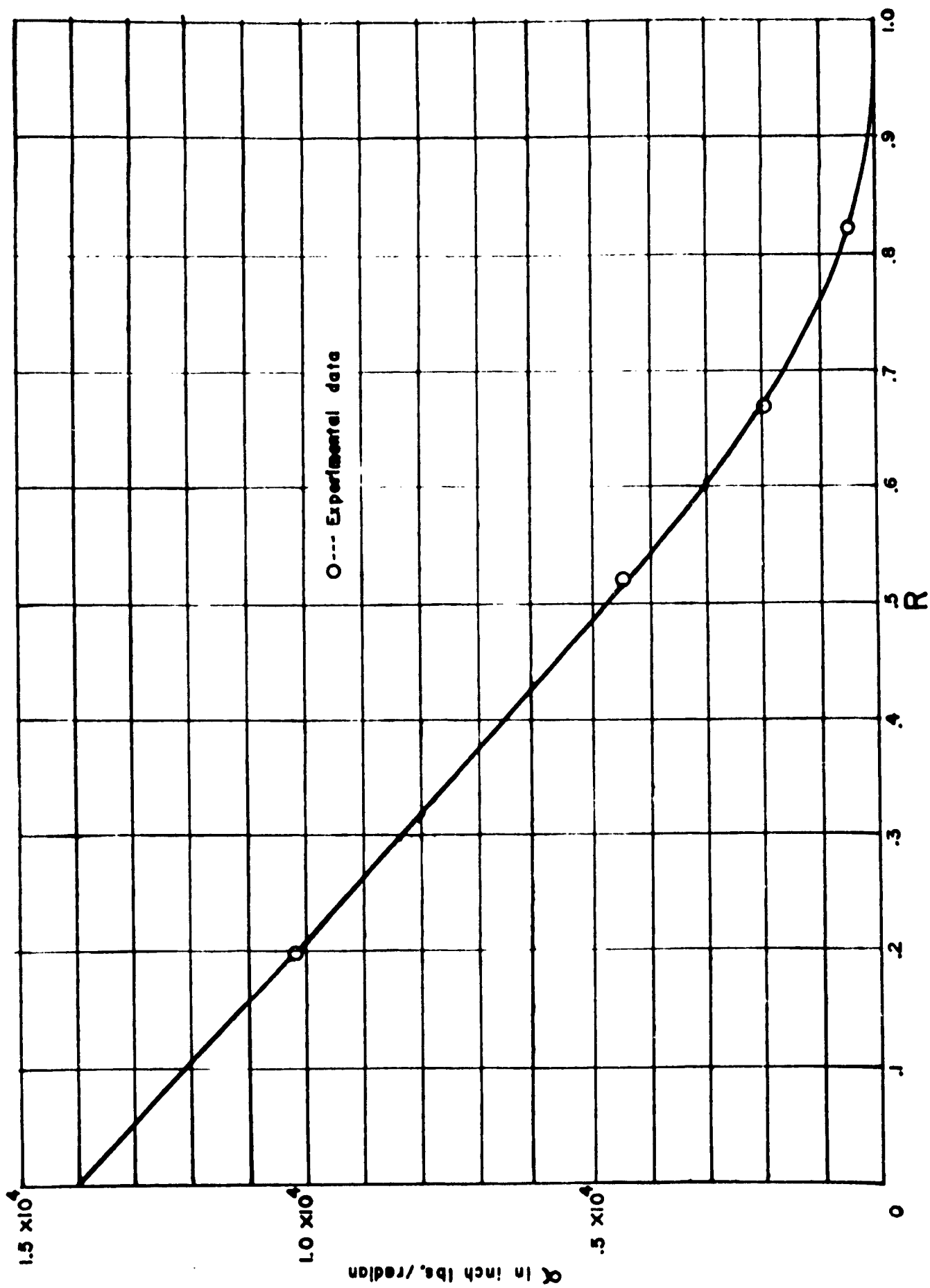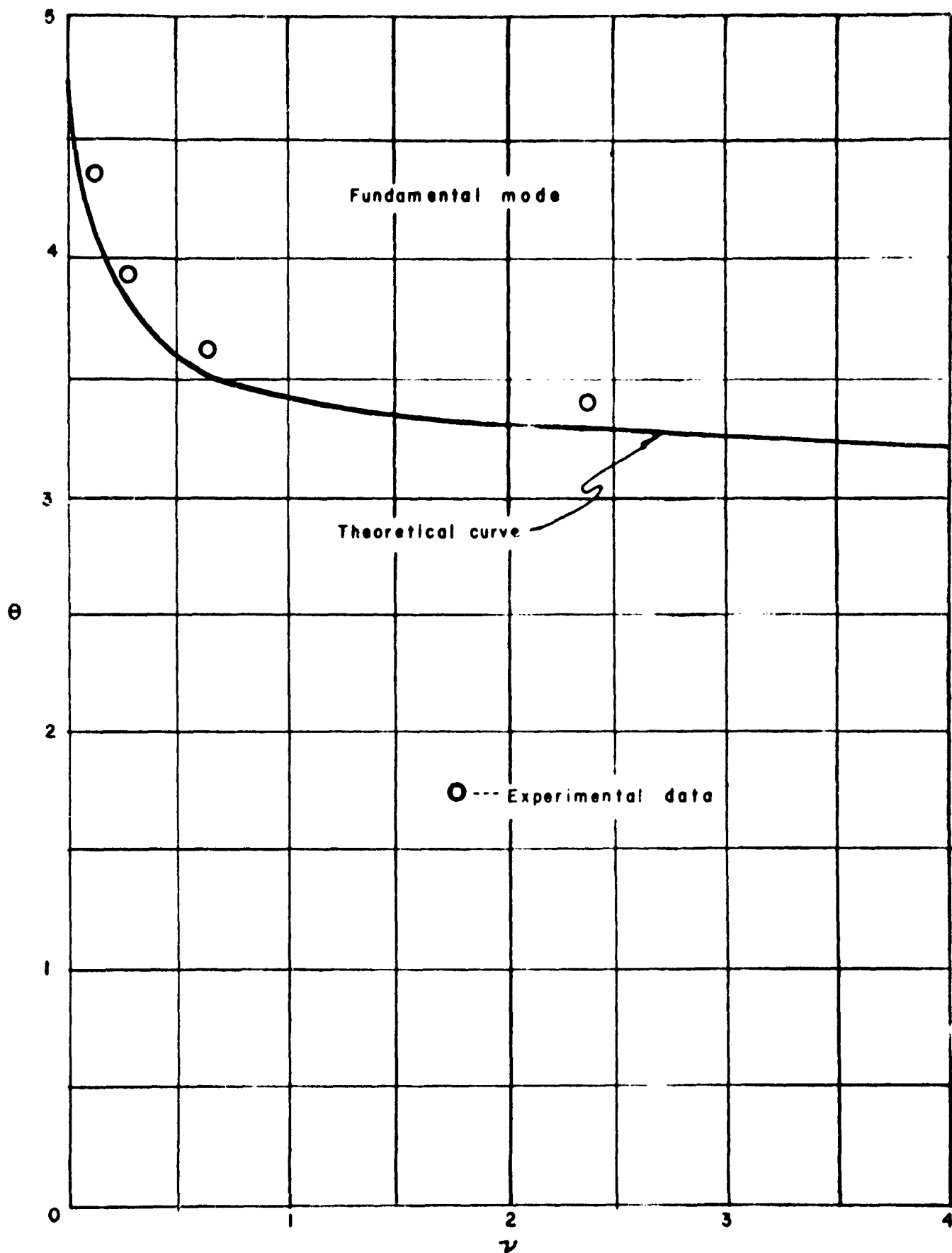
Figure 8 Varition of root of frequency equation Θ with stiffness ratio

Fundamental mode

O--- Experimental data

Circular frequency $p = \dfrac{c}{l^2}\sqrt{\dfrac{Eh^2}{12\rho}}$

Circular frequency $p = \frac{c}{a^2}\sqrt{\frac{D}{\rho h}} = \frac{c}{a^2}\sqrt{\frac{Eh^2}{12\rho(1-\nu^2)}}$

O --- Experimental data

Fundamental mode of vibration
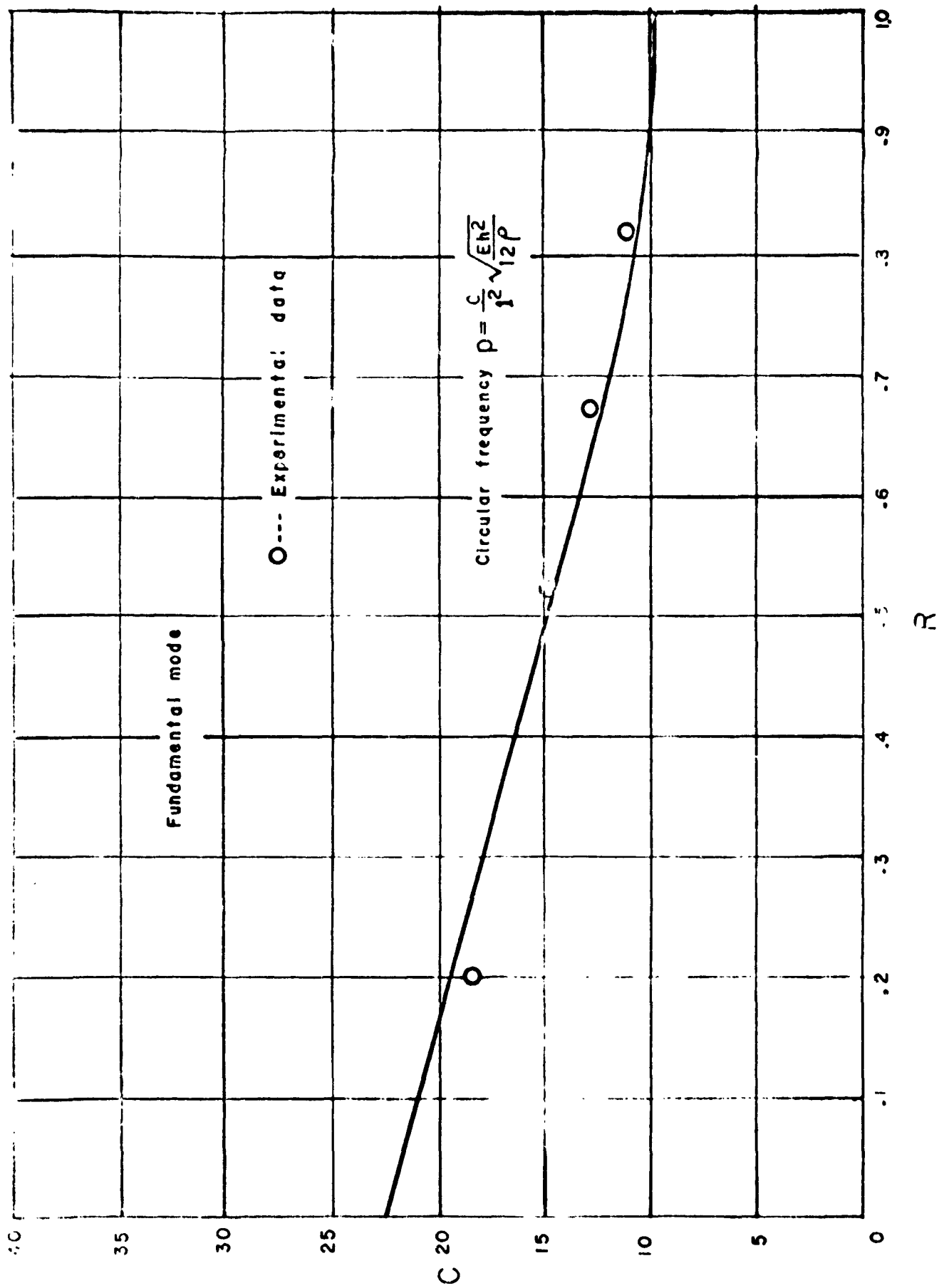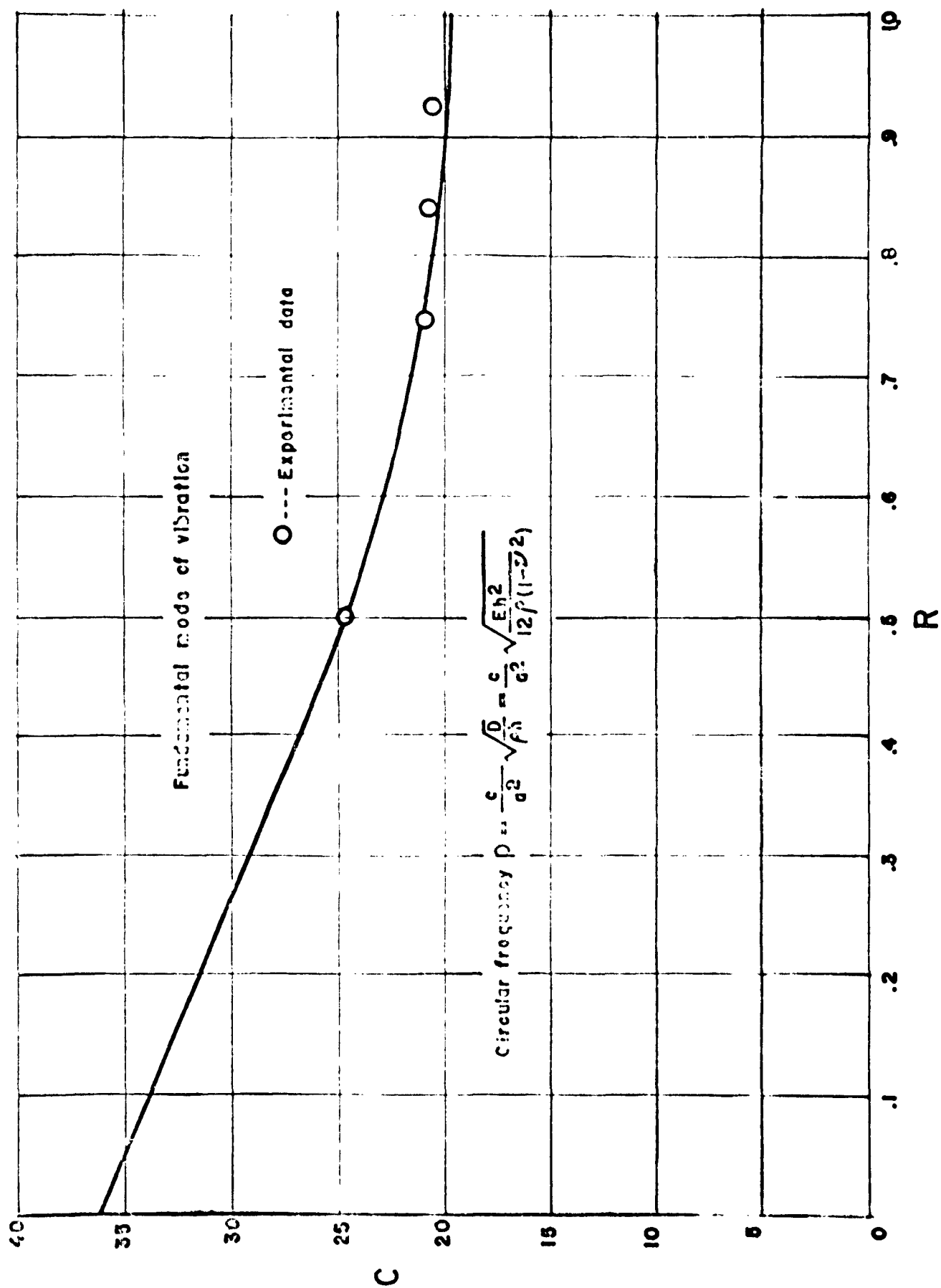
Figure 10 Variation of frequency coefficient C with notch ratio R for plate.

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